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Effect of anisotropy on inertio-elastic instability of rotating disks

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Abstract

In the classical approach of determining the stresses and displacements due to a centrifugal force in rotating disks, the inertia force considered does not include the radial displacement thus yielding stable solutions. Inclusion of the displacement in the centrifugal force results in instability at certain rotational speeds. The present study addresses the problem of instability in rotating polar-orthotropic disks. Following a brief outline of the classical analysis, the stress redistribution solutions are presented. The solutions are obtained in terms of non-dimensional parameters. The parameter defined as the ratio of circumferential stiffness to radial stiffness has been found to have the most considerable effect on the critical rotational speed. Various cases are considered, and the corresponding critical rotational parameters are presented in tables. The comparison of stresses obtained from the classical approach with the redistributed stresses is displayed graphically. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The existence of a static inertio-elastic instability of rotating isotropic disks has been first noticed by Brunelle (1971). The problem of instability is not crucial for rotating structural elements made of usual metals, but it is of primary importance in rotating structures made of low modulus, high yield strength materials. Polymeric materials which may display anisotropic properties are examples of such materials.

Theoretical modeling of the stress distribution available in the literature due to centrifugal forces in rotating plates includes planar elasticity and laminated plate theories. Studies by Bert and Niedenfuhr (1963), Reddy and Srinath (1973), Chang (1975) and Genta and Gola (1981) dealt with determining stresses via an elasticity approach in orthotropic single-ply circular plates with the outer boundary free of any constraints. Bert (1975) used a laminated plate theory on layered plates of rectangular anisotropy with extension-bending coupling and with stress-free boundaries by which approximate solutions were obtained. Christensen and Wu (1977) determined the optimal shapes of orthotropic flywheels using a uniform strain

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failure criterion. Uniform strain meant variable thickness along the radial direction. Tutuncu (1995) examined the effect of anisotropy on the centrifugal stress distribution in laminated rotating disks for various outer boundary conditions. The laminates considered were specially orthotropic. When the rotating plate is restrained at the outer edge, compressive stresses begin to occur near the outer boundary. If the rotational speed reaches a critical value, these stresses will cause local buckling. This problem was addressed by Mostaghel and Tadjbakhsh (1973) for the isotropic case and by Tutuncu and Durdu (1998) for the specially orthotropic case.

In all the works cited above, which will be termed as the classical approach from here on, the centrifugal force has been assumed to be $\rho\omega^2 r$, where ρ is the mass per unit area, ω is the rotational speed and r is the radial coordinate. The classical approach does not predict any instability so long as the circular plate rotates freely without any constraint along its outer boundary. However, including the actual centrifugal force $\rho\omega^2(r+u)$ where u is the radial displacement, results in a stress redistribution along the radius and predicts instability at a critical value of ω. The analysis along with a comparison with the results of the classical approach is given in the following sections.

The rotating composite disk in question displays polar orthotropic material characteristics. Polar orthotropy is achieved when the polar coordinate axes are also the axes of material symmetry. The freely rotating plate is either a full plate or is fixed to a rigid shaft of radius a.

2. Classical approach

Consider a circular plate of radius R and uniform thickness h which is rotating with the angular speed ω about an axis perpendicular to its plane. Assuming steady rotation, symmetric deformation, and no bending, displacement field takes the form

$$u = u(r), \qquad v = v(\theta) = 0, \qquad w = w(r) = 0. \tag{1}$$

From here on, u and r will denote the non-dimensional quantities defined as u/R and r/R. Because of the axisymmetry of the problem, all the stress and strain components are independent of the θ coordinate. The in-plane strain components are

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \gamma_{r\theta} = 0. \tag{2}$$

Constitutive equations for the specially orthotropic plate take the following form:

$$\begin{Bmatrix} N_r \\ N_\theta \\ N_{r\theta} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ 0 \end{Bmatrix}, \tag{3}$$

where N_r and N_θ are, respectively, the stresses in the r and θ directions. Subscripts 1 and 2 refer, respectively, to the r and θ directions.

The only non-trivial equilibrium equation is in the radial direction

$$\frac{\mathrm{d}N_r}{\mathrm{d}r} + \frac{N_r - N_\theta}{r} + \rho\omega^2 r = 0. \tag{4}$$

Substituting Eq. (2) into Eq. (3) and thus obtaining expressions for N_r and N_θ in terms of stiffness and radial displacement, the equilibrium equation can be used to derive the governing equation of radial displacement as

$$r^2 u'' + r u' - \lambda^2 u = -\Omega^2 r^3, \tag{5}$$

where $\lambda^2 = A_{22}/A_{11},\ \Omega^2 = \rho\omega^2 R^3/A_{11},\ r = r/R,\ u = u/R$ as defined previously and $(\ )'$ refers to differentiation with respect to r. The solution of Eq. (5) is

$$u = C_1 r^{\lambda} + C_2 r^{-\lambda} + \frac{\Omega^2 r^3}{\lambda^2 - 9} \quad (\lambda^2 \neq 9), \tag{6}$$

$$u = D_1 r^3 + D_2 r^{-3} + \frac{(1 - 6\log r)}{36}\Omega^2 r^3 \quad (\lambda^2 = 9). \tag{7}$$

Using Eqs. (2) and (3) along with the above displacement equations, the stresses are expressed as follows:
For $\lambda^2 \neq 9$,

$$\frac{N_r}{A_{11}} = C_1(\lambda + v)r^{\lambda-1} - C_2(\lambda - v)r^{-(\lambda+1)} + \frac{(3 + v)r^2\Omega^2}{\lambda^2 - 9}, \tag{8}$$

$$\frac{N_\theta}{A_{11}} = C_1\lambda(\lambda + v)r^{\lambda-1} + C_2\lambda(\lambda - v)r^{-(\lambda+1)} + \frac{(\lambda^2 + 3v)r^2\Omega^2}{\lambda^2 - 9} \tag{9}$$

and for $\lambda^2 = 9$,

$$\frac{N_r}{A_{11}} = D_1(3 + v)r^2 - D_2(3 - v)r^{-4} + \frac{(v - 3 - 6(v + 3)\log r)r^2\Omega^2}{36}, \tag{10}$$

$$\frac{N_\theta}{A_{11}} = D_1(\lambda^2 + 3v)r^2 + D_2(\lambda^2 - 3v)r^{-4} + \frac{(\lambda^2 - 3v - 6(\lambda^2 + 3v)\log r)r^2\Omega^2}{36}, \tag{11}$$

where $v = A_{12}/A_{11}$ which essentially represents Poisson's effect.

In the case of a full plate ($a/R = 0$), the condition $u(0) = 0$ demands that $C_2 = D_2 = 0$ and the stress-free outer boundary means $N_r = 0$ at $r = 1$ thus yielding the following expressions for C_1 and D_1:

$$C_1 = \frac{(3 + v)\Omega^2}{(\lambda + v)(9 - \lambda^2)}, \tag{12}$$

$$D_1 = \frac{(3 - v)\Omega^2}{36(3 + v)}. \tag{13}$$

If the plate is fixed to a rigid shaft of radius a, the boundary conditions are that there is no radial displacement at $r = a/R$ and that $N_r = 0$ at $r = 1$. Under these conditions, the constants C_1, C_2, D_1 and D_2 are determined as

$$C_1 = \frac{\left[\left(\frac{a}{R}\right)^{\lambda+3}(v - \lambda) - v - 3\right]\Omega^2}{(\lambda^2 - 9)\left[\left(\frac{a}{R}\right)^{2\lambda}(\lambda - v) + \lambda + v\right]}, \tag{14}$$

$$C_2 = \frac{\left[\left(\frac{a}{R}\right)^{2\lambda}(v + 3) - \left(\frac{a}{R}\right)^{\lambda+3}(v + \lambda)\right]\Omega^2}{(\lambda^2 - 9)\left[\left(\frac{a}{R}\right)^{2\lambda}(\lambda - v) + \lambda + v\right]}, \tag{15}$$

$$D_1 = \frac{(3 - v)\left[\left(\frac{a}{R}\right)^6\left(6\log\frac{a}{R} - 1\right) + 1\right]\Omega^2}{36\left[\left(\frac{a}{R}\right)^6(3 - v) + 3 + v\right]}, \tag{16}$$

$$D_2 = \frac{\left(\frac{a}{R}\right)^6 \left[(v+3)\log\frac{a}{R} - 1\right]\Omega^2}{6\left[\left(\frac{a}{R}\right)^6 (3-v) + 3 + v\right]}. \tag{17}$$

It should be noted that no instability is present in the classical results for any value of the rotational parameter Ω.

3. Stress redistribution and instability

The actual centrifugal force $\rho\omega^2(r+u)$ will now be considered. The strain–displacement and constitutive equations remain unchanged while the equilibrium equation takes the form

$$\frac{\mathrm{d}N_r}{\mathrm{d}r} + \frac{N_r - N_\theta}{r} + \rho\omega^2(r+u) = 0. \tag{18}$$

Following the same steps as in the classical approach, the displacement equilibrium equation is obtained as

$$r^2u'' + ru' + (\Omega^2r^2 - \lambda^2)u = -\Omega^2r^3. \tag{19}$$

The solution of Eq. (19) can be obtained in the form of Bessel functions. Two cases must be considered: (a) λ is integer and (b) λ is real. For $\lambda =$ integer,

$$u(r) = c_1J_\lambda(\Omega r) + c_2Y_\lambda(\Omega r) + A\left[J_\lambda(\Omega r)\int_{r_0}^{r} z^2Y_\lambda(\Omega z)\mathrm{d}z - Y_\lambda(\Omega r)\int_{r_0}^{r} z^2J_\lambda(\Omega z)\mathrm{d}z\right] \tag{20}$$

and for $\lambda =$ real,

$$u(r) = d_1J_\lambda(\Omega r) + d_2J_{-\lambda}(\Omega r) + B\left[J_\lambda(\Omega r)\int_{r_0}^{r} z^2J_{-\lambda}(\Omega z)\mathrm{d}z - J_{-\lambda}(\Omega r)\int_{r_0}^{r} z^2J_\lambda(\Omega z)\mathrm{d}z\right], \tag{21}$$

where $A = (\pi\Omega^2/2)$, $B = -(\pi\Omega^2/2\sin\pi\lambda)$ and J_λ, Y_λ, respectively, are Bessel functions of the first and second kind. The constants c_1, c_2, d_1 and d_2 are to be determined from the appropriate boundary conditions. Using Eqs. (2) and (3) along with Eqs. (20) and (21), the stresses are expressed as follows:

For $\lambda =$ integer,

$$\begin{aligned}\frac{N_r}{A_{11}} &= \left[c_1 + A\int_{r_0}^{r} z^2Y_\lambda(\Omega z)\mathrm{d}z\right]\left[\Omega J_\lambda'(\Omega r) + \frac{v}{r}J_\lambda(\Omega r)\right] \\ &\quad + \left[c_2 - A\int_{r_0}^{r} z^2J_\lambda(\Omega z)\mathrm{d}z\right]\left[\Omega Y_\lambda'(\Omega r) + \frac{v}{r}Y_\lambda(\Omega r)\right],\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{N_\theta}{A_{11}} &= \left[c_1 + A\int_{r_0}^{r} z^2Y_\lambda(\Omega z)\mathrm{d}z\right]\left[v\Omega J_\lambda'(\Omega r) + \frac{\lambda^2}{r}J_\lambda(\Omega r)\right] \\ &\quad + \left[c_2 - A\int_{r_0}^{r} z^2J_\lambda(\Omega z)\mathrm{d}z\right]\left[v\Omega Y_\lambda'(\Omega r) + \frac{\lambda^2}{r}Y_\lambda(\Omega r)\right]\end{aligned} \tag{23}$$

and for $\lambda =$ real,

$$\begin{aligned}\frac{N_r}{A_{11}} &= \left[d_1 + B\int_{r_0}^{r} z^2J_{-\lambda}(\Omega z)\mathrm{d}z\right]\left[\Omega J_\lambda'(\Omega r) + \frac{v}{r}J_\lambda(\Omega r)\right] \\ &\quad + \left[d_2 - B\int_{r_0}^{r} z^2J_\lambda(\Omega z)\mathrm{d}z\right]\left[\Omega J_{-\lambda}'(\Omega r) + \frac{v}{r}J_{-\lambda}(\Omega r)\right],\end{aligned} \tag{24}$$

$$\frac{N_\theta}{A_{11}} = \left[d_1 + B\int_{r_0}^{r} z^2 J_{-\lambda}(\Omega z)\mathrm{d}z\right]\left[v\Omega J'_\lambda(\Omega r) + \frac{\lambda^2}{r}J_\lambda(\Omega r)\right]$$
$$+\left[d_2 - B\int_{r_0}^{r} z^2 J_{\lambda}(\Omega z)\mathrm{d}z\right]\left[v\Omega J'_{-\lambda}(\Omega r) + \frac{\lambda^2}{r}J_{-\lambda}(\Omega r)\right], \tag{25}$$

where $J'_\lambda(\Omega r) = \frac{1}{2}J_{\lambda-1}(\Omega r) - \frac{1}{2}J_{\lambda+1}(\Omega r)$ and $Y'_\lambda(\Omega r) = \frac{1}{2}Y_{\lambda-1}(\Omega r) - \frac{1}{2}Y_{\lambda+1}(\Omega r)$ (Abromowitz and Stegun, 1972).

For a full plate ($r_0 = 0$) with $\lambda =$ integer, $u(0) = 0$ demands that $c_2 = 0$, and along the boundary ($r = 1$) the stress-free condition $N_r(1) = 0$ yields

$$c_1 = A\left(\frac{a_{22}}{a_{21}}\int_0^1 z^2 J_\lambda(\Omega z)\,\mathrm{d}z - \int_0^1 z^2 Y_\lambda(\Omega z)\,\mathrm{d}z\right), \tag{26}$$

when $\lambda =$ real, $u(0) = 0$ gives $b_2 = 0$ and $N_r(1) = 0$ gives

$$d_1 = B\left(\frac{b_{22}}{b_{21}}\int_0^1 z^2 J_\lambda(\Omega z)\,\mathrm{d}z - \int_0^1 z^2 J_{-\lambda}(\Omega z)\,\mathrm{d}z\right), \tag{27}$$

where

$$a_{21} = \Omega J'_\lambda(\Omega) + vJ_\lambda(\Omega), \qquad a_{22} = \Omega Y'_\lambda(\Omega) + vY_\lambda(\Omega), \qquad b_{21} = a_{21}, \qquad b_{22} = \Omega J'_{-\lambda}(\Omega) + vJ_{-\lambda}(\Omega).$$

In the case of a plate fixed to a rigid shaft ($r_0 = a/R$), for $\lambda =$ integer, applying the conditions $u(a/R) = 0$ and $N_r(1) = 0$ yields

$$c_1 = \frac{-a_{12}f}{a_{11}a_{22} - a_{12}a_{21}}, \tag{28}$$

$$c_2 = \frac{a_{11}f}{a_{11}a_{22} - a_{12}a_{21}}, \tag{29}$$

and for $\lambda =$ real, the constants are determined as

$$d_1 = \frac{-b_{12}g}{b_{11}b_{22} - b_{12}b_{21}}, \tag{30}$$

$$d_2 = \frac{b_{11}g}{b_{11}b_{22} - b_{12}b_{21}}, \tag{31}$$

where

$$a_{11} = J_\lambda\left(\Omega\frac{a}{R}\right), \qquad a_{12} = Y_\lambda\left(\Omega\frac{a}{R}\right), \qquad b_{11} = a_{11}, \qquad b_{12} = J_{-\lambda}\left(\Omega\frac{a}{R}\right),$$

$$f = A\left(a_{22}\int_{a/R}^1 z^2 J_\lambda(\Omega z)\mathrm{d}z - a_{12}\int_{a/R}^1 z^2 Y_\lambda(\Omega z)\mathrm{d}z\right),$$
$$g = B\left(b_{22}\int_{a/R}^1 z^2 J_\lambda(\Omega z)\mathrm{d}z - b_{12}\int_{a/R}^1 z^2 J_{-\lambda}(\Omega z)\mathrm{d}z\right).$$

Note that in both cases, at a certain speed, the denominators in the constants given above will go to zero, thus, resulting in unbounded displacements and stresses. With the constants now fully determined, the displacements and the corresponding stresses are readily calculated using Eqs. (20) and (21) and Eqs. (22)–(25).

4. Results

Using the stress redistribution expressions the critical rotational parameter at which instability occurs will now be determined for various cases. There are essentially three parameters affecting the critical parameter Ω_{cr} namely, v, λ and a/R. Table 1 shows the effect of λ and a/R on the critical parameter for $v = 0.3$. The ratio $a/R = 0$ corresponds to the full plate. The degree of anisotropy is given by λ and $\lambda = 1$ corresponds to the isotropic case. For the isotropic case, the results given in the present study exactly match those of Brunelle (1971). As a/R increases, the effect of λ on the critical rotational parameter decreases. For high values of a/R, the effect of anisotropy is virtually negligible. Poisson's effect on Ω_{cr} is displayed in Table 2. The case for $a/R = 0.1$ is chosen, and it is clearly noticed that v has minimal effect which further decreases with increasing λ. The range of values of v are determined arbitrarily due to the fact that the open interval $(0, \frac{1}{2})$ for Poisson's number in isotropic materials do not necessarily constitute the bounds for v in anisotropic materials.

Since the critical speed is most strongly affected by the degree of anisotropy, the parameter λ^2, which is the ratio of circumferential stiffness to radial stiffness, must be reserved as a design parameter in applications. Increasing λ increases the stability of the rotating disk. A comparison of the stresses determined by the classical analysis with the redistributed stresses is given in Fig. 1. These stresses are calculated at

Table 1
Effect of λ and a/R on the critical rotational parameter

a/R	Ω_{cr}			
$v = 0.3$	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 3.0$
0.0	0.7593	1.5788	2.2480	4.0473
0.1	1.6078	2.0879	2.6469	4.3395
0.2	1.8272	2.2039	2.6967	4.3417
0.3	2.1089	2.4043	2.8212	4.3610
0.4	2.4841	2.7135	3.0552	4.4371
0.5	3.0088	3.1834	3.4541	4.6397
0.6	3.7954	3.9237	4.1286	5.0913
0.7	5.1055	5.1944	5.3393	6.0613
0.8	7.7245	7.7795	7.8703	8.3433
0.9	15.579	15.605	15.648	15.875

Table 2
Effect of λ and v on the critical rotational parameter

v	Ω_{cr}			
$a/R = 0.1$	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 3.0$
0.0	1.3517	1.8793	2.4677	4.2012
0.1	1.4453	1.9537	2.5308	4.2489
0.2	1.5302	2.0231	2.5904	4.2950
0.3	1.6078	2.0879	2.6469	4.3395
0.4	1.6792	2.1488	2.7007	4.3825
0.5	1.7453	2.2060	2.7517	4.4241
0.6	1.8068	2.2599	2.8003	4.4643
0.7	1.8641	2.3109	2.8467	4.5032
0.8	1.9178	2.3591	2.8909	4.5409
0.9	1.9681	2.4047	2.9331	4.5773
1.0	2.0155	2.4481	2.9735	4.6126

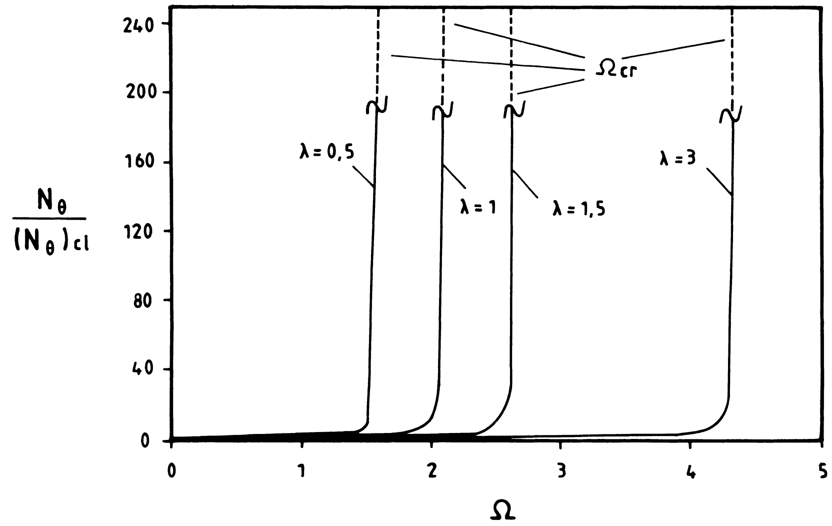

Fig. 1. Hoop stress ratio vs. the rotational parameter.

$a/R = 0.1$ for $v = 0.3$. The difference between the two gets substantially large as the critical rotational parameter is approached. The stabilizing effect of increased stiffness ratio is once again demonstrated.

5. Conclusions

Stresses in rotating polar-orthotropic disks have been calculated using the classical approach which does not include the radial displacement in the centrifugal force expression, and no instability has been noted in the results. Considering the actual centrifugal force with the radial displacement resulted in redistributed stresses and instability has occurred at a certain critical rotational speed. The results have been obtained as functions of non-dimensional parameters, namely, the ratio of inner radius to outer radius, the ratio of circumferential stiffness to radial stiffness which gives the degree of anisotropy, and Poisson's term. Full circular plates and plates fixed to a rigid shaft are considered. Obviously, the full plate possesses the smallest critical speed while the stability of the plates fixed to a rigid shaft is increased by increasing the ratio of inner radius to outer radius. The most considerable effect on the stability has been found to be due to the stiffness ratio; increasing the circumferential stiffness with respect to the radial stiffness contributed an increase to the stability of the rotating plate. Poisson's term has remained to be a minor factor in the stability analysis. Being restricted by the geometric properties, the designer has the flexibility of choosing a material with a suitable stiffness ratio.

References

Abromowitz, M., Stegun, I.A., 1972. Handbook of Mathematical Functions. Dover Publications, New York.

Bert, C.W., 1975. Centrifugal stresses in arbitrarily laminated rectangular anisotropic circular discs. J. Strain Anal. 10 (2), 84–92.

Bert, C.W., Niedenfuhr, F.W., 1963. Stretching of a polar-orthotropic disk of varying thickness under arbitrary body forces. AIAA Journal 1 (6), 1385–1390.

Brunelle, E.J., 1971. Stress redistribution and instability of rotating beams and disks. AIAA Journal 9 (4), 758–759.

Chang, C.I., 1975. The anisotropic rotating disks. Int. J. Mech. Sci. 17 (4), 397–402.

Christensen, R.M., Wu, E.M., 1977. Optimal design of anisotropic (fiber-reinforced) flywheels. J. Comp. Mater. 11, 395–404.
Genta, G., Gola, M., 1981. The stress distribution in orthotropic rotating discs. J. Appl. Mech. 48, 559–562.
Mostaghel, N., Tadjbakhsh, I., 1973. Buckling of rotating rods and plates. Int. J. Mech. Sci. 15 (5), 429–437.
Reddy, T.Y., Srinath, H., 1973. Elastic stresses in a rotating anisotropic annular disk of variable thickness and variable density. Int. J. Mech. Sci. 16 (2), 85–89.
Tutuncu, N., 1995. Effect of anisotropy on stresses in rotating discs. Int. J. Mech. Sci. 37 (8), 873–881.
Tutuncu, N., Durdu, A., 1998. Determination of buckling speed for rotating orthotropic disk restrained at outer edge. AIAA Journal 36 (1), 89–93.